Semilattice Polymorphisms and Chordal Graphs

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Abstract

We will review some basic properties of semilattice orders, and semilattice polymorphisms, and see an algebraic characterization of chordal graphs. In particular, we will see that a graph G is chordal if and only if it has a semilattice polymorphism such that G is a subgraph of the comparability graph of the semilattice.

1 Semilattices and Polymorphisms

In this section we will take a first look at definitions related to semilattices and polymorphisms, and see some of their basic properties. Note, however, that as we will be applying them to finite graphs, all sets and graphs are assumed to be finite.

Definition. A partial order on a set V is called a *semilattice order* if for every two elements $u, v \in V$ there exists a unique greatest lower bound $glb(u, v)$.
A binary function $\wedge : V \times V \to V : (u, v) \mapsto u \wedge v$ is called a *semilattice function* if it satisfies idempotency, associativity, and commutativity conditions.

Observe that given a semilattice order on a set V, a semilattice function on V may be defined as $u \wedge v = glb(u, v)$. On the other hand, given a semilattice function $\wedge$ a semilattice order on V can be defined as $u \leq v \iff u \wedge v = u$. So, the two definitions given above are equivalent, and we will use them interchangably throughout this paper. We will also use standard definitions and notations of partial orders, total orders on a set, and also those of graphs.

Notation. For two comparable elements $u, v \in V$, equipped with a semilattice order, we will use the notation $u \parallel v$, and for u, v noncomparable we will use $u \perp v$.

Definition. For a given semilattice order on a set V, define the *comparability graph* of $\leq$ to be the graph with vertex set V, where there is an edge between

$u \neq v$ if $u \perp v$. Also, for $u, v \in V$ we say v *covers* u if $v \geq u$ and there exists no x with $v > x > u$. Now, we define the *cover diagram*, or the *Hasse diagram* of $\leq$ to be the digraph with vertex set V, where there is an arc from v to u if v covers u.

Note that the cover diagram of a partial order cannot have a directed cycle, as it follows from existence of a directed cycle $v_1 \to v_2 \to \ldots \to v_k \to v_1$, v_i's distinct, that $v_1 > v_2 > \ldots > v_k > v_1$, a contradiction. However, it is legitimate for the underlying simple graph of a cover diagram to have a cycle, as in the cover diagram in Figure 1:

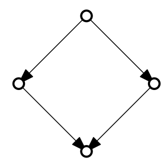

Figure 1: An example of a cover diagram

In what follows, we will be studying semilattice orders on graphs, thus we will find it useful for the graphs to be assumed to have a loop at each vertex, so we make it a running assumption throughout the paper.

Another important tool when studying semilattice orders on graphs are polymorphisms on graphs, whose definition is given below:

Definition. A *polymorphism* on a graph G is a d-ary function $f : V(G)^d \to V(G)$, satisfying

$$u_i \sim v_i, i = 1, \ldots, d \Rightarrow f(u_1, \ldots, u_d) \sim f(v_1, \ldots, v_d).$$

An *SL polymorphism* on a graph G is a semilattice function on $V(G)$ which is also a polymorphism.

Note that a semilattice on G is an SL polymorphism iff it satisfies

$$a \sim a', b \sim b' \Rightarrow (a \wedge b) \sim (a' \wedge b'),$$

which will be referred to as *polymorphism property*, whose two of implications is stated in the proposition below:

Proposition 1 *[1] The polymorphism property implies*

(i) **(The min property)** $(u \geq v \geq w, u \sim w) \Rightarrow v \sim w$*,*

(ii) **(The V-property)** $u \sim v \Rightarrow u \sim (u \wedge v)$.

Proof.

(i) Suppose $u \geq v \geq w$, and $u \sim w$, and let $a = u, a' = w$ and $b = b' = v$; we have $a \sim a'$, $b \sim b'$, which together with the polymorphism property implies
$$v = u \wedge v = a \wedge b \sim a' \wedge b' = w \wedge v = w.$$

(ii) Suppose $u \sim v$, and let $a = a' = b = u$ and $b' = v$. Again we have $a \sim a'$, $b \sim b'$, thus
$$u = u \wedge u = a \wedge b \sim a' \wedge b' = u \wedge v.$$

□

Note that the converse of the above proposition does not necessarily hold, namely the min property and the V-property do not necessarily imply the polymorphism property, as in the case where $a \parallel a', b \parallel b'$, and $a \wedge b \parallel a' \wedge b'$. Knowing that, we start looking for a third property whose combination with the min property and the V-property implies the polymorphism property. However, before moving in that direction it is helpful to notice that we can restrict our attention to connected graphs:

Lemma 1 *[1] A graph G admits an SL polymorphism if and only if each component of G does.*

Proof. Let G admit an SL polymorphism $\wedge$, and G_1 be a component of G. To see that the restriction of $\wedge$ to G_1 results in a semilattice order (and thus an SL polymorphism), let $u, v \in G_1$, we show that $u \wedge v$ also needs to be in G_1. Take a vertex $w \in G_1$, and two w-paths of length k, $w = x_1 \sim x_2 \sim \ldots \sim x_k = u$, and $w = y_1 \sim y_2 \sim \ldots \sim y_k = v$. (Note that G_1 having loops at each node allows us to assume the two paths have the same length.) Now applying $\wedge$ to these paths, we get
$$w = w \wedge w = x_1 \wedge y_1 \sim x_2 \wedge y_2 \sim \ldots \sim x_k \wedge y_k = u \wedge v,$$
so $u \wedge v \in G_1$.

For the backward direction let G_1, G_2 be two components of G, equipped with SL polymorphisms $\leq_1, \leq_2$, respectively, m_1 the minimum element of $\leq_1$ and M_2 any maximal element of $\leq_2$, and define $\leq$ on $G_1 \cup G_2$ to be $\leq_1$ on G_1, $\leq_2$ on G_2, $m_1 \geq M_2$, and take the transitive closure. Since $\leq$ is an extension of $\leq_i$ on G_i, $i = 1, 2$, it is clear that it satisfies polymorphism property at each G_i. So it suffices to show that when $a \sim a'$ in G_1 and $b \sim b'$ in G_2, then $a \wedge b \sim a' \wedge b'$, which can be easily verified, as
$$a \wedge b = M_2 \wedge b \sim M_2 \wedge b' = a' \wedge b'.$$
The lemma now follows by induction on the number of components of G. □

In light of the lemma above, from now on we assume the graphs to be connected.

2 Tree Semilattices

Recall that the underlying simple graph of a cover diagram can contain cycles, as in Figure 1. However, as we will see shortly, disregarding such cover diagrams enables us to obtain a characterization of SL polymorphisms.

Definition. A semilattice is called a *tree semilattice* if its cover diagram is a tree, or, equivalently, if $u \geq v, w$ implies $v \perp w$. An SL polymorphism in which the semilattice is a tree semilattice is called a *tree SL polymorphism.*

Lemma 2 *[1] For a tree semilattice order on a set A, and $a, a', b, b' \in A$ we have:*

(i) If $a \geq a', b \geq b'$, and $a' \parallel b'$, then $a \wedge b = a' \wedge b'$.

(ii) If $a \wedge b \parallel a' \wedge b'$, then $a \wedge b \geq a \wedge a' = b \wedge b' \leq a' \wedge b'$.

(iii) If $a \wedge b \geq a' \wedge b'$, then either $a' \wedge b' = a' \wedge a = a' \wedge b$, or $b' \wedge a' = b' \wedge a = b' \wedge b$.

Proof.

(i) If $a \geq a', b \geq b'$, then $a \wedge b \geq a \wedge b' \geq a' \wedge b'$. To see $a' \wedge b' \geq a \wedge b$, note that $a \geq a', a \wedge b$, so $a' \perp a \wedge b$. If $a \wedge b \geq a'$, then $b \geq a \wedge b \geq a'$ together with $b \geq b'$ imlies $a' \perp b'$, contradicting $a' \parallel b'$, so $a' \geq a \wedge b$, and similarly $b' \geq a \wedge b$, so $a' \wedge b' \geq a \wedge b$.

(ii) it follows from $a \geq a \wedge a', a \wedge b$ that $a \wedge a' \perp a \wedge b$. Now if it is not the case that $a \wedge b \geq a \wedge a'$, then the part of the cover diagram containing $a, a \wedge b, a \wedge a'$ contains a cycle. In the same way it follows that $a' \wedge b' \geq b \wedge b'$.

(iii) Similar to the proof of part (ii).

□

Now we have the necessary tools to provide a necessary and sufficient condition for a tree semilattice to be a tree SL polymorphism.

Proposition 2 *[1] A tree semilattice on a graph G is a tree SL polymorphism iff it satisfies the min property, the V-property, and the polymorphism property for all edges aa', bb' with $a \parallel a', b \parallel b', a \wedge a' = b \wedge b'$.*

Proof. The necessity of the conditions follows immediately from Proposition 1 and the lemma above. For their sufficiency, take edges aa', bb' of G; we need to show that $a \wedge b \sim a' \wedge b'$. If $a \wedge b = a' \wedge b'$ then the conclusion is trivial. Further, if $a \wedge b \parallel a' \wedge b'$, then it must be the case that $a \parallel a', b \parallel b'$, as $\leq$ is a tree semilattice.

It now follows from part (*ii*) of the lemma above that $a \wedge a' = b \wedge b'$, therefore, by the premise of the proposition, it satisfies the polymorphism property.

So, without loss of generality, we may assume that $a \wedge b > a' \wedge b'$. Now, from part (*iii*) of the previous lemma we may assume that $a' \wedge b' = a \wedge a'$, which, together with the V-property, implies that $a' \wedge b' \sim a$. Now, given the min property and $a \geq a \wedge b \geq a' \wedge b'$, we have $a \wedge b \sim a' \wedge b'$, as desired. □

3 SL Polymorphisms and Chordal Graphs

In this section we will see how we can use tree SL polymorphisms to get a characterization for chordal graphs.

Definition. A graph G is called *chordal* if each of its cycles of length greater than 3 has a chord.

Note that earlier we assummed our graphs to have a loop at each node; however, for the purposes of this section, we consider a graph to be chordal if it is chordal after removing the loops.

It is well known, see e.g. [2], that a graph is chordal iff it has a *perfect elimination order (PEO)*, where a PEO is a total order on $V(G)$ such that

$$(u \leq v \leq w), (u \sim w), (v \sim w) \Rightarrow u \sim v.$$

Here, we see another characterization of chordal graphs using SL polymorphisms.

Definition. An SL polymorphism on a graph G is called *non-crossing* if G is a subgraph of the comparability graph of $\leq$.

Theorem 1 *[1] For a graph G the following are equivalent:*

(i) G is chordal.

(ii) G admits a non-crossing SL polymorphism.

(iii) G admits a non-crossing tree SL polymorphism.

Proof. Observe that the implication (*iii*) $\Rightarrow$ (*ii*) is immediate. Now, we will see (*ii*) $\Rightarrow$ (*i*), and (*i*) $\Rightarrow$ (*iii*).

(*ii*) $\Rightarrow$ (*i*) Suppose G has a non-crossing SL polymorphism, and C is a cycle in G. As C is a subgraph of the comparability graph of $\leq$, there must be a maximal vertex of C, greater than both its neighbours in C, which are adjacent by the polymorphism property . So, if C has length > 3, then it has a chord.

$(i) \Rightarrow (iii)$ Suppose G is chordal, and take a PEO of G, $\leq$. We will modify $\leq$ to a tree semilattice $\leq'$ of G through the following steps:

1. Let $\leq$-minimum vertex be the $\leq'$-minimum.
2. For other vertices, say v, let the maximum neighbour below it in $\leq$ be covered by v in $\leq'$.
3. Extend $\leq'$ transitively.

We show that $\leq'$ is a non-crossing tree SL polymorphism of G. By its construction, it is clearly a tree ordering. To see that G is non-crossing, for a contradiction, suppose it is not, and consider all edges of G whose two endpoints are parallel with respect to $\leq'$, and let u be the minimal vertex with respect to $\leq'$ that is an endpoint of such an edge uv. Without loss of generality, we may assume $u \geq v$. As u does not cover v, it covers some other neighbour u' such that $v \leq u'$, and so, as $\leq$ is a PEO, $u' \sim v$. But now $v \leq' u'$ contradicts $v \not\leq' u$, and $v \parallel' u'$ contradicts minimality of u.

Now it remains for us to show that $\leq'$ satisfies the polymorphism property. Exploiting proposition 2, and noting that for non-crossing tree semilattices, all the premises of the proposition, except for the min property, trivially hold, we will verify the min property. Let $u \leq' v \leq' w$, and $u \sim w$, we need to show that $u \sim v$. Since $v \leq' w$, there exists a path $v = v_1 \sim v_2 \sim \ldots \sim v_k = w$, where $v_{i-1} \leq' v_i, i = 2, \ldots, k$. We have

$$u \leq' v_{k-1} \leq' v_k, v_k \sim v_{k-1}, v_k \sim u,$$

and these relations also hold with respect to $\leq$, as $\leq$ is a linear extension of $\leq'$. Since $\leq$ is a PEO, $u \sim v_{k-1}$. It now follows inductively that $u \sim v_1 = u$, as desired.

□

References

[1] P. Hell, M. Siggers, *Semilattice polymorphisms and chordal graphs*, European J. Combin. 36 (2014), 694-706.

[2] D. B. West, *Introduction to graph Theory*, Prentice Hall, 2 edition, (September 2000).